

MOTIVES OF DELIGNE-MUMFORD STACKS

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ABSTRACT. For every smooth and separated Deligne-Mumford stack F , we associate a motive $M(F)$ in Voevodsky's category of mixed motives with rational coefficients $\mathbf{DM}^{eff}(k, \mathbb{Q})$. When F is proper over a field of characteristic 0, we compare $M(F)$ with the Chow motive associated to F by Toen ([34]). Without the properness condition we show that $M(F)$ is a direct summand of the motive of a smooth quasi-projective variety.

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1. INTRODUCTION

The study of Deligne-Mumford stacks from a motivic perspective began in [7] where the notion of the *DMC*-motive associated to a proper and smooth

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Deligne-Mumford stack was introduced as a tool for defining Gromov-Witten invariants. The construction of the category \mathcal{M}_k^{DM} of DMC -motives uses A^* -Chow cohomology theories for Deligne-Mumford stacks as described in [16, 23, 22, 26, 36, 13]. These A^* -Chow cohomology theories coincide with rational coefficients. In [34, Theorem 2.1], Toen shows that the canonical functor $\mathcal{M}_k \rightarrow \mathcal{M}_k^{DM}$, from the category of usual Chow motives, is an equivalence rationally. In particular, to every smooth and proper Deligne-Mumford stack M , Toen associates a Chow motive $h(M)$.

In this paper we construct motives for smooth (but not necessarily proper) Deligne-Mumford stacks as objects of Voevodsky's triangulated category of motives $\mathbf{DM}^{eff}(k, \mathbb{Q})$. In the proper case, we compare these motives with the Chow motives we get using Toen's equivalence of categories $\mathcal{M}_k \simeq \mathcal{M}_k^{DM}$. Without assuming properness, our construction of the motive of a smooth Deligne-Mumford stack F seems to be the first one. However, in [15, Thm 0.1] Gillet and Soulé constructed a motivic invariant attached to F , namely a complex of Chow motives. We hope to recover their invariant by applying Bondarko's weight functor to $M(F)$. (See [10, Prop. 6.3.1].) We leave this for a future investigation.

The paper is organised as follows.

In section 2, we briefly review Morel-Voevodsky \mathbb{A}^1 -homotopy category $\mathbf{H}^{\acute{e}t}(k)$ and Voevodsky triangulated category of motives $\mathbf{DM}^{eff}(k, \mathbb{Q})$. We also construct the functor $M : \mathbf{H}^{\acute{e}t}(k) \rightarrow \mathbf{DM}^{eff}(k, \mathbb{Q})$. Given a presheaf of small groupoids F , we associate an object $Sp(F)$ in $\mathbf{H}^{\acute{e}t}(k)$. The motive of F is defined to be $M(F) := M(Sp(F))$. We then show that for a Deligne-Mumford stack F and an étale atlas $u : U \rightarrow F$ we have an isomorphism $M(U_{\bullet}) \cong M(F)$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Here U_{\bullet} is the Čech hypercovering corresponding to the atlas $u : U \rightarrow F$.

In section 3, we compare the motive of a separated Deligne-Mumford stack F with the motive of the coarse moduli space of F . If $\pi : F \rightarrow X$ is the coarse moduli space of a separated Deligne-Mumford stack F , we show that the natural morphism $M(F) \rightarrow M(X)$ is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. We then prove projective bundle formula and blow-up formula for smooth Deligne-Mumford stacks. We also construct the Gysin triangle associated to a smooth, closed substack Z of a smooth Deligne-Mumford stack F .

In section 4, we show that for any smooth and separated Deligne-Mumford stack F over a field of characteristic zero, the motive $M(F)$ is a direct factor of the motive of a smooth quasi-projective scheme. If F is proper we may take this scheme to be projective.

In section 5, we show that the motivic cohomology of a Deligne-Mumford stack (see [22, 3.0.2]) is representable in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.

Finally in section 6 we compare our construction with Toen's construction and prove that for any smooth and proper Deligne-Mumford stack F there is a canonical isomorphism $\iota \circ h(F) \cong M(F)$; this is Theorem 6.4. Here

$\iota : \mathcal{M}_k^{\text{eff}} \rightarrow \mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ is the fully faithful embedding described in [29, Proposition 20.1].

The paper ends with two appendixes. In Appendix A, we show that some naturally defined functor $\omega : \mathcal{M}_k^{\text{eff}} \rightarrow PSh(\mathcal{V}_k)$ is fully faithful (see A.1). This statement appears without proof in [32, 2.2] and is mentioned in [34, page12]. It is also needed in the proof of 6.4. In Appendix B we provide a technical result used in Appendix A.

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2. THE GENERAL CONSTRUCTION

In this section we describe our construction of the motive associated to a smooth Deligne-Mumford stack. In fact, our construction applies more generally to any stack but the existence of atlases can be used to give explicit models. We start by recalling the motivic categories used in this paper.

2.1. Review of motivic categories. Let Sm/k be the category of smooth separated finite type k -schemes and denote $PSh(Sm/k)$ the category of presheaves of sets on Sm/k . Also denote $\Delta^{\text{op}}PSh(Sm/k)$ the category of spaces, i.e., presheaves of simplicial sets. As usual Δ is the category of simplices.

$\Delta^{\text{op}}PSh(Sm/k)$ has a local model structure with respect to the étale topology [20, Theorem 2.4 and Corollary 2.7]. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y} \in \Delta^{\text{op}}PSh(Sm/k)$ is a local weak equivalence if the induced morphisms on the stalks (for the étale topology) are weak equivalences of simplicial sets. Cofibrations are monomorphisms and fibrations are characterised by the right lifting property. We denote by $\mathbf{H}_s^{\text{ét}}(k)$ the homotopy category of $\Delta^{\text{op}}PSh(Sm/k)$ with respect to the étale local model structure, i.e., obtained by inverting formally the local weak equivalences.

Following [30, §3.2], we consider the Bousfield localisation of the local model structure on $\Delta^{\text{op}}PSh(Sm/k)$ with respect to the class of maps $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$ where $\mathcal{X} \in \Delta^{\text{op}}PSh(Sm/k)$. The resulting model structure will be simply called the (étale) motivic model structure. The homotopy category with respect to this étale motivic model structure is denoted by $\mathbf{H}^{\text{ét}}(k)$. (We warn the reader that in [30, §3.2] the Nisnevich topology is used instead of the étale topology.)

Remark 2.1. Denote $(\Delta^{\text{op}}PSh(Sm/k))_{\bullet}$ the category of pointed spaces, i.e., presheaves of pointed simplicial sets on Sm/k . We also have the pointed versions of the local and motivic model structures where weak equivalences

are detected after forgetting the pointing. The homotopy categories are denoted by $\mathbf{H}_{s,\bullet}^{\text{ét}}(k)$ and $\mathbf{H}_{\bullet}^{\text{ét}}(k)$ respectively.

Now we briefly recall some facts on Voevodsky's motives. Recall that $\text{SmCor}(k)$ is the category of finite correspondences. Objects of this category are smooth k -schemes X . For $X, Y \in \text{Sm}/k$, $\text{Hom}_{\text{SmCor}(k)}(X, Y)$ is given by the group of finite correspondences $\text{Cor}(X, Y)$. This is the free abelian group generated by integral closed subschemes $W \subset X \times Y$ which are finite and surjective on a connected component of X . Thus, if $X = \coprod_i X_i$ we have $\text{Cor}(X \times Y) = \bigoplus_i \text{Cor}(X_i \times Y)$.

A presheaf with transfers is a contravariant additive functor on $\text{SmCor}(k)$. Denote by $PST(k, \mathbb{Q})$ the category of presheaves with transfers with values in the category of \mathbb{Q} -vector spaces. A typical example is given by $\mathbb{Q}_{tr}(X)$ for $X \in \text{Sm}/k$. This presheaf associates to each $U \in \text{Sm}/k$ the vector space $\text{Cor}(U, X) \otimes \mathbb{Q}$.

Analogous to the local and motivic model category structures on the category of spaces, we have a local and a motivic model structures on the category $K(PST(k, \mathbb{Q}))$ of complexes of presheaves with transfers. A morphism $K \rightarrow L$ in $K(PST(k, \mathbb{Q}))$ is an étale weak equivalence if it induces quasi-isomorphisms on stalks for the étale topology (or the Nisnevich topology; it doesn't matter in the presence of transfers). Cofibrations are monomorphisms and fibrations are characterized by the right lifting property. This gives the local model category structure on $K(PST(k, \mathbb{Q}))$ (cf. [4, Theorem 2.5.7]). The homotopy category is nothing but the derived category of étale sheaves with transfers $\mathbf{D}(\text{Str}(\text{Sm}/k, \mathbb{Q}))$. (Here $\text{Str}(\text{Sm}/k, \mathbb{Q})$ is the category of étale sheaves with transfers). The motivic model structure is the Bousfield localisation of the local model structure with respect to the class of morphisms $\mathbb{Q}_{tr}(\mathbb{A}^1 \times X)[n] \rightarrow \mathbb{Q}_{tr}(X)[n]$ for $X \in \text{Sm}/k$ and $n \in \mathbb{Z}$. The resulting homotopy category with respect to the motivic model structure is denoted by $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$. This is Voevodsky's triangulated category of mixed motives (with rational coefficients).

The functor $\mathbb{Q}_{tr}(-) : \text{Sm}/k \rightarrow PST(k, \mathbb{Q})$ extends to a functor $\mathbb{Q}_{tr} : PSh(\text{Sm}/k) \rightarrow PST(k, \mathbb{Q})$ given by $\mathbb{Q}_{tr}(F) = \text{Colim}_{X \rightarrow F} \mathbb{Q}_{tr}(X)$ for any presheaf of sets F on Sm/k . In the next statement, $N(-)$ denotes the functor that associates the normalized chain complex to a simplicial object in an additive category (cf. [17, page 145]).

Proposition 2.2. *There exists a functor $M : \mathbf{H}_{\bullet}^{\text{ét}}(k) \rightarrow \mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$. It sends a simplicial scheme X_{\bullet} to the $N\mathbb{Q}_{tr}(X_{\bullet})$.*

Proof. This is well known. We give a sketch of proof here. The functor $\mathbb{Q}_{tr}(-) : PSh(\text{Sm}/k) \rightarrow PST(k, \mathbb{Q})$ extends to a functor

$$N\mathbb{Q}_{tr}(-) : \Delta^{\text{op}} PSh(\text{Sm}/k) \rightarrow K(PST(k, \mathbb{Q})).$$

There is a functor $\Gamma : K(PST(k, \mathbb{Q})) \rightarrow \Delta^{\text{op}} PSh(\text{Sm}/k)$ right adjoint to $N\mathbb{Q}_{tr}$ (cf. [17, page 149]). We will show that the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen

adjunction for the projective motivic model structures. (These model structures are different from the ones described above: they have the same weak equivalences but the cofibrations in the projective ones are defined by the left lifting property with respect to section-wise trivial fibrations of presheaves of simplicial sets and surjective morphisms of complexes of presheaves with transfers respectively.)

The functor Γ takes section-wise weak equivalences in $K(PST(k, \mathbb{Q}))$ to section-wise weak equivalences in $\Delta^{op}PSh(Sm/k)$ and it takes surjective morphisms to section-wise fibrations in $\Delta^{op}PSh(Sm/k)$. Hence for the projective global model structures the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen adjunction. By [12, Theorem 6.2] the projective étale local model structure on $\Delta^{op}PSh(Sm/k)$ is the Bousfield localisation of the global projective model structure with respect to general hypercovers for the étale topology. Let S be the class of those hypercovers. To show that the pair $(N\mathbb{Q}_{tr}, \Gamma)$ is a Quillen adjunction for the étale local model structures, we need to show that the left derived functor of $N\mathbb{Q}_{tr}$ maps morphisms in S to étale local weak equivalences in $K(PST(k, \mathbb{Q}))$. For this it is enough to show that Γ maps a local fibrant object C_\bullet of $K(PST(k, \mathbb{Q}))$ to an S -local object of $\Delta^{op}PSh(Sm/k)$. Showing that $\Gamma(C_\bullet)$ is S -local is equivalent to showing that the étale hypercohomology ${}_{\acute{e}t}^n(X, C_\bullet)$ is isomorphic to $H^n(C_\bullet(X))$ for any $X \in Sm/k$ and $n \geq 0$. Now, the hypercohomology ${}_{\acute{e}t}^n(X, C_\bullet)$ can be calculated using Čech hypercovers $U_\bullet \rightarrow X$ and moreover by [29, Prop 6.12] the complex $\mathbb{Q}_{tr}(U_\bullet)$ is a resolution of the étale sheaf $\mathbb{Q}_{tr}(X)$. Since C_\bullet is local fibrant we have

$$H^n(C_\bullet(X)) = Hom_{Ho(K(PST(k, \mathbb{Q})))}(\mathbb{Q}_{tr}(U_\bullet), C_\bullet[n]) = H^n(Tot(C_\bullet(U_\bullet))).$$

(Here $Ho(K(PST(k, \mathbb{Q})))$ is the homotopy category with respect to global projective model structure on $K(PST(k, \mathbb{Q}))$.) Now passing to the colimit over hypercovers $U_\bullet \rightarrow X$ we get ${}_{\acute{e}t}^n(X, C_\bullet) \cong H^n(C_\bullet(X))$.

At this point we get a functor $\mathbf{H}_s^{\acute{e}t}(k) \rightarrow \mathbf{D}(Str(Sm/k))$ and it remains to show that this functor takes the maps $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$ to motivic weak equivalences. This is clear by construction. \square

Remark 2.3. *There is also a functor $M : \mathbf{H}^{\acute{e}t}(k) \rightarrow \mathbf{DM}^{eff, \acute{e}t}(k, \mathbb{Q})$ to Voevodsky's category of étale motives with integral coefficients. It is constructed exactly as above. (Note that with integral coefficients, the categories of étale and Nisnevich motives are different: we denote them $\mathbf{DM}^{eff, \acute{e}t}(k, \mathbb{Q})$ and $\mathbf{DM}^{eff}(k, \mathbb{Q})$ respectively; with rational coefficient these categories are the same.)*

2.2. The construction. Let $Grpd$ be the category of (small) groupoids. Let \mathcal{C} be any category.

Consider $2 - Fun(\mathcal{C}^{op}, Grpd)$ the category of lax 2-functors from \mathcal{C} to $Grpd$. Recall that a lax 2-functor F associates to $X \in \mathcal{C}$ a groupoid $F(X)$, to $f : Y \rightarrow X$ a functor $F(f) : F(X) \rightarrow F(Y)$, and to composable morphisms f and g an isomorphism $F(f) \circ F(g) \cong F(g \circ f)$. The 1-morphisms between two

lax 2-functors \mathcal{F} and \mathcal{G} are lax natural transformations H such that for any $f : Y \rightarrow X \in \mathcal{C}$ there is a natural isomorphism between the functors $G(f) \circ H_X$ and $H_Y \circ F(f)$. For any composable morphisms f and g , we have the usual compatibility conditions. 2-isomorphisms between lax transformations H and H' are given by isomorphisms of functors $a_X : H_X \cong H'_X$ for each $X \in \mathcal{C}$, such that for any $f : Y \rightarrow X$ we have $G(f)(a_X) = a_Y(F(f))$.

For objects $X, Y \in \mathcal{C}$, consider the set $\text{Hom}_{\mathcal{C}}(Y, X)$ as a discrete groupoid, i.e, all morphisms are identities. In this way, the functor $\text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \rightarrow \text{Grpd}$ is a strict 2-functor which we denote by $h(X)$.

Lemma 2.4. *Let $F \in 2-\text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd})$. There is a surjective equivalence of categories $\text{Hom}_{2-\text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd})}(h(X), F) \rightarrow F(X)$ given by evaluating at $\text{id}_X \in h(X)(X)$.*

Proof. Given any lax natural transformation $H : h(X) \rightarrow F$ we get an object $X' := H_X(\text{id}_X) \in F(X)$. Given two lax natural transformations H, H' and a 2-isomorphism a between them, we get an isomorphism $a_X(\text{id}_X) : H_X(\text{id}_X) \cong H'_X(\text{id}_X)$. Let $X' \in F(X)$. We have a natural transformation given by $G_Y(f : Y \rightarrow X) = F(f)(X')$. Since F is a lax presheaf we have $F(f \circ g)(X') \cong F(g) \circ F(f)$ for any $Z \xrightarrow{g} Y \xrightarrow{f} X$. Hence we get the required natural transformation between $F(g) \circ G_Y$ and $G_Z \circ h(X)(g)$. Moreover let $H, G : h(X) \rightarrow F$ such that there exists a morphism $f : H_X(\text{id}_X) \rightarrow G_X(\text{id}_X) \in F(X)$. We define a unique 2-isomorphism a between H and G in the following way. For any $g \in h(X)(Y)$ we have $H_Y(g) \cong F(g)(H_X(\text{id}_X))$ given by the structure of the lax natural transformation. Similarly we get $G_Y(g) \cong F(g)(G_X(\text{id}_X))$. But then there exists $F(g)(f) : F(g)(H_X(\text{id}_X)) \cong F(g)(G_X(\text{id}_X))$. So $a_Y(g) : H_Y(g) \cong G_Y(g)$ and $a_X(\text{id}_X) : H_X(\text{id}_X) \rightarrow G_X(\text{id}_X)$ is equal to f . \square

Remark 2.5. *In general $\text{Hom}_{2-\text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd})}(h(X), F)$ is not small unless \mathcal{C} is small. Let Sch/k be the category of finite type k -schemes. We fix $\mathcal{C} \subset \text{Sch}/k$ which is a full small subcategory equivalent to Sch/k . For any $X \in \text{Sch}/k$ and $F \in 2-\text{Fun}((\text{Sch}/k)^{\text{op}}, \text{Grpd})$ the association $X \mapsto \text{Hom}_{2-\text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd})}(h(X)|_{\mathcal{C}}, F|_{\mathcal{C}})$ gives a strict presheaf of groupoids. We denote it by $h_{\text{st}}(F)$. By 2.4 we have an equivalence $F|_{\mathcal{C}} \cong h_{\text{st}}(F)|_{\mathcal{C}}$.*

Definition 2.6. *Let $F \in 2-\text{Fun}((\text{Sch}/k)^{\text{op}}, \text{Grpd})$. Then the \mathbb{A}^1 -homotopy type of F is the space $\text{Sp}(F) := \text{Ner}(h_{\text{st}}(F))|_{S^m/k}$ considered as an object of $\mathbf{H}^{\text{ét}}(k)$. Here Ner is the nerve functor.*

Definition 2.7. *Let $F \in 2-\text{Fun}((\text{Sch}/k)^{\text{op}}, \text{Grpd})$. Then the motive of F is defined as $M(F) := M(\text{Sp}(F))$. This gives a functor*

$$M : 2-\text{Fun}((\text{Sch}/k)^{\text{op}}, \text{Grpd}) \rightarrow \mathbf{DM}^{\text{eff}}(k, \mathbb{Q}).$$

Using 2.3 we can also define an integral version of the motive of F , which we also denote $M(F)$ if no confusion can arise.

2.3. The case of a Deligne-Mumford stack. Let $F : (\text{Sch}/k)^{\text{op}} \rightarrow \text{Grpd}$ be a lax 2-functor.

Definition 2.8. *The functor F is a stack in the étale topology if it satisfies the following axioms where $\{f_i : U_i \rightarrow U\}_{i \in I}$ is an étale covering of $U \in \text{Sch}/k$ and $f_{ij,i} : U_i \times_U U_j \rightarrow U_i$ are the projections.*

- (1) (*Glueing of morphisms*) If X and Y are two objects of $F(U)$, and $\phi_i : F(f_i)(X) \cong F(f_i)(Y)$ are isomorphisms such that $F(f_{ij,i})(\phi_i) = F(f_{ij,j})(\phi_j)$, then there exists an isomorphism $\eta : X \cong Y$ such that $F(f_i)(\eta) = \phi_i$.
- (2) (*Separation of morphisms*) If X and Y are two objects of $F(U)$, and $\phi : X \cong Y$, $\psi : X \cong Y$ are isomorphisms such that $F(f_i)(\phi) = F(f_i)(\psi)$, then $\phi = \psi$.
- (3) (*Glueing of objects*) If X_i are objects of $F(U_i)$ and $\phi_{ij} : F(f_{ij,j})(X_j) \cong F(f_{ij,i})(X_i)$ are isomorphisms satisfying the cocycle condition
$$(F(f_{ijk,ij})(\phi_{ij})) \circ (F(f_{ijk,jk})(\phi_{jk})) = F(f_{ijk,ik})(\phi_{ik}),$$
then there exist an object X of $F(U)$ and $\phi_i : F(f_i)(X) \cong X_i$ such that $\phi_{ji} \circ (F(f_{ij,i})(\phi_i)) = F(f_{ij,j})(\phi_j)$.

Remark 2.9. There is a notion of strict stacks (see [19], [21]). If F is a strict presheaf of groupoids then by [21, lemma 7, lemma 9] there exists a strict stack $St(F)$ and a morphism $st : F \rightarrow St(F)$ such that st is a local weak equivalence, i.e., st induces equivalences of groupoids on stalks. The stack $St(F)$ is called the associated stack of F and the functor is called the stackification functor.

For any groupoid object $R \rightrightarrows U$ in Sch/k we can associate a strict presheaf of groupoids $h(R \rightrightarrows U)$ (see the proof of lemma 2.12).

Definition 2.10. A Deligne-Mumford stack F is a stack on Sch/k admitting a local equivalence (stalk-wise equivalence in the étale topology) $h(R \rightrightarrows U) \rightarrow F$, where $R \rightrightarrows U$ is a groupoid object in Sch/k , such that both morphisms $R \rightarrow U$ are étale and $R \rightarrow U \times_F U$ is finite.

Remark 2.11. Our definition of a Deligne-Mumford stack is equivalent to that of separated finite type Deligne-Mumford stack from [28]. The morphism $p : U \rightarrow F$ is representable and is called the atlas of F . We also have $R \cong U \times_F U$. We say that F is smooth if U is smooth.

Given an atlas $f : U \rightarrow F$ of a smooth Deligne-Mumford stack F , we get a simplicial object U_\bullet in Sm/k by defining $U_i = U \times_F \cdots \times_F U$ ($i+1$ times) and the face and degeneracy maps are defined by relative diagonal and partial projections.

Lemma 2.12. For $R \rightrightarrows U$ as above we have $\text{Ner}(h(R \rightrightarrows U)) = U_\bullet$.

Proof. By definition we have $\text{Ob}(h(R \rightrightarrows U)(S)) = \text{Hom}_{\text{Sm}/k}(S, U)$ and $\text{Mor}(h(R \rightrightarrows U)(S)) = \text{Hom}_{\text{Sm}/k}(S, R)$. The set of two composable morphisms in $h(R \rightrightarrows U)(S)$ is $(R \times_U R)(S)$ where $R \times_U R$ is the fiber product

of the maps $s : R \rightarrow U$ and $t : R \rightarrow U$. More generally the set of n -composable morphisms in $h(R \rightrightarrows U)(S)$ is $R \times_U R \times_U \cdots \times_U R$ (n times). Since $R \cong U \times_F U$ and the maps s and t are first and second projections respectively we have

$$(Ner(h(R \rightrightarrows U)))_n(S) = \underbrace{(U \times_F U) \times_U \cdots \times_U (U \times_F U)}_{n \text{ times}}$$

which is isomorphic to U_n . \square

Theorem 2.13. *Let $U \rightarrow F$ be an atlas for a smooth Deligne-Mumford stack F . There is a canonical étale local weak equivalence $U_\bullet \rightarrow Sp(F)$.*

Proof. We know that $h(R \rightrightarrows U)$ is locally weakly equivalent to F . Hence the morphism $Ner(h(R \rightrightarrows U)) \rightarrow Sp(F)$ is a local weak equivalence. The claim follows now from lemma 2.12. \square

Corollary 2.14. *Let $U \rightarrow F$ be an atlas for a smooth Deligne-Mumford stack F . The canonical map $M(U_\bullet) \rightarrow M(F)$ in $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ is an isomorphism. (This is also true integrally.)*

Proof. This follows from proposition 2.2 and theorem 2.13. \square

Let $F' \rightarrow F$ be a morphism of strict presheaves of groupoids. Let F'_\bullet be the simplicial presheaf of groupoids such that $F'_i := F' \times_F F' \times_F \cdots \times_F F'$ ($i+1$ times). Let $Ner(F'_\bullet)$ be the bi-simplicial presheaf such that $Ner(F'_\bullet)_{\bullet, i} := Ner(F'_i)$. Let $\text{diag}(Ner(F'_\bullet))$ be the diagonal.

Lemma 2.15. *Let $p : F' \rightarrow F$ be an étale, representable, surjective morphism of Deligne-Mumford stacks (here stacks are strict presheaves of groupoids). Then the canonical morphism*

$$\text{diag}(Ner(F'_\bullet)) \rightarrow Ner(F)$$

is an étale local weak equivalence.

Proof. Let $U \rightarrow F$ be an atlas and U_\bullet be the associated Čech simplicial scheme. Let $U'_{\bullet, \bullet}$ be the bi-simplicial algebraic space such that $U'_{\bullet, i} := U_\bullet \times_F F'_i$ for $i \geq 0$. Hence, $U'_{j, \bullet} := U_j \times_F F'_\bullet$ for $j \geq 0$. There are natural morphisms $\text{diag}(U'_{\bullet, \bullet}) \rightarrow \text{diag}(Ner(F'_\bullet))$ and $\text{diag}(U'_{\bullet, \bullet}) \rightarrow U_\bullet$. For $i, j \geq 0$, $U'_{\bullet, i} \rightarrow F'_i$ and $U'_{j, \bullet} \rightarrow U_j$ are étale Čech hypercovering, hence $U'_{\bullet, i} \rightarrow Ner(F'_i)$ and $U'_{j, \bullet} \rightarrow U_j$ are étale local weak equivalences. By [11, XII.3.3]

$$\text{diag}(U'_{\bullet, \bullet}) \cong \text{hocolim}_{n \in \Delta}(U'_{\bullet, n}) \cong \text{diag}(Ner(F'_\bullet))$$

and

$$\text{diag}(U'_{\bullet, \bullet}) \cong \text{hocolim}_{n \in \Delta}(U'_{n, \bullet}) \cong U_\bullet.$$

This proves the lemma. \square

3. MOTIVES OF DELIGNE-MUMFORD STACKS, I

In this section, we first show that the motive of a separated Deligne-Mumford stack is naturally isomorphic to the motive of its coarse moduli space. We also prove blow-up and projective bundle formulas for smooth Deligne-Mumford stacks. We end the section with the construction of the Gysin triangle associated with a smooth closed substack Z of a smooth Deligne-Mumford stack F .

3.1. Motive of coarse moduli space. Let F be a separated Deligne-Mumford stack. A coarse moduli space for F is a map $\pi : F \rightarrow X$ to an algebraic space such that π is initial among maps from F to algebraic spaces, and for every algebraically closed field k the map $[F(k)] \rightarrow X(k)$ is bijective (where $[F(k)]$ denotes the set of isomorphism classes of objects in the small category $F(k)$). If F is a separated Deligne-Mumford stack over a field k of characteristic 0, then a coarse moduli space $\pi : F \rightarrow X$ exists.

Let X be a scheme and let G be a group acting on X . Then G acts on the presheaf $\mathbb{Q}_{tr}(X)$. Let $\mathbb{Q}_{tr}(X)_G$ be the G -coinvariant presheaf, such that for any $Y \in Sm/k$ we have $\mathbb{Q}_{tr}(X)_G(Y) := (\mathbb{Q}_{tr}(X)(Y))_G$.

Lemma 3.1. *Let X be a smooth quasi-projective scheme and let G be a finite group acting on X . Let $[X/G]$ be the quotient Deligne-Mumford stack and X/G be the quotient scheme. Then*

- (1) $M([X/G]) \cong \mathbb{Q}_{tr}(X)_G$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$;
- (2) $\mathbb{Q}_{tr}(X/G) \cong \mathbb{Q}_{tr}(X)_G$ as presheaves.

Hence the canonical morphism $M([X/G]) \rightarrow \mathbb{Q}_{tr}(X/G)$ is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.

Proof. To deduce (1), we observe that the morphism $X \rightarrow [X/G]$ sending X to the trivial G -torsor $X \times G \rightarrow X$ is an étale atlas. Let X_\bullet be the corresponding Čech simplicial scheme. Then $\mathbb{Q}_{tr}(X_\bullet) \cong M([X/G])$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Moreover, $\mathbb{Q}_{tr}(X_\bullet)(Y) \cong (\mathbb{Q}_{tr}(X)(Y) \otimes \mathbb{Q}[EG])/G$. Hence the complex $\mathbb{Q}_{tr}(X_\bullet)(Y)$ computes the homology of G with coefficient in the G -module $\mathbb{Q}_{tr}(X_\bullet)(Y)$. Since G is finite and we work with rational coefficients, we have $\mathbb{Q}_{tr}(X_\bullet)(Y) \cong (\mathbb{Q}_{tr}(X)(Y))_G$ in the derived category of chain complexes of \mathbb{Q} -vector spaces.

To deduce (2), we observe that the canonical quotient morphism $\pi : X \rightarrow X/G$ is finite and surjective. Let m be the generic degree of π , then the morphism $\Gamma_\pi : \mathbb{Q}_{tr}(X) \rightarrow \mathbb{Q}_{tr}(X/G)$ has a section $\frac{1}{m}{}^t\Gamma_\pi$. Hence $\mathbb{Q}_{tr}(X/G)$ is isomorphic to the image of the projector $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi$. But $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi = \frac{1}{|G|} \sum_{g \in G} g$ whose image is isomorphic to $\mathbb{Q}_{tr}(X)_G$. \square

Remark 3.2. *In the proof above, the composition $\frac{1}{m}{}^t\Gamma_\pi \circ \Gamma_\pi$ is well defined (cf. [29, Definition 1A.11]). Indeed, since X/G is normal, the finite correspondence ${}^t\Gamma_\pi$ is a relative cycle over X/G by [29, Theorem 1A.6].*

Theorem 3.3. *Let F be a separated smooth Deligne-Mumford stack over a field k of characteristic 0. Let $\pi : F \rightarrow X$ be the coarse moduli space. Then the natural morphism $M(\pi) : M(F) \rightarrow \mathbb{Q}_{tr}(X)$ is an isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$.*

Proof. By [35, Prop 1.17] and [36, Prop 2.8], there exists an étale covering $(U_i)_{i \in I}$ of X , such that $U_i \cong X_i/H_i$ and $F_i := U_i \times_X F \cong [X_i/H_i]$ for quasi-projective smooth schemes X_i and finite groups H_i . Let $F' := \coprod F_i$ and $X' := \coprod X_i/H_i$. Then by lemma 2.15, $M(diag(Ner(F'_\bullet))) \cong M(F)$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Similarly, $\mathbb{Q}_{tr}(X'_\bullet) \cong \mathbb{Q}_{tr}(X)$. To show that $M(F) \cong \mathbb{Q}_{tr}(X)$, it is then enough to show that $M(Ner(F'_n)) \cong M(X'_n)$. Hence we are reduced to the case $F = [X/G]$ which follows from lemma 3.1. \square

3.2. Motive of a projective bundle. Let \mathcal{E} be a vector bundle of rank $n+1$ on a smooth finite type Deligne-Mumford stack F and let $Proj(\mathcal{E})$ denote the associated projective bundle over F .

Theorem 3.4. *There exists a canonical isomorphism in $\mathbf{DM}^{eff}(k, \mathbb{Q})$:*

$$M(Proj(\mathcal{E})) \rightarrow \bigoplus_{i=0}^n M(F) \otimes \mathbb{Q}(i)[2i].$$

Proof. Let $a : U \rightarrow F$ be an atlas of F and $V := Proj(a^*(\mathcal{E})) \rightarrow Proj(\mathcal{E})$ be the induced atlas of $Proj(\mathcal{E})$.

The line bundle $\mathcal{O}_{Proj(\mathcal{E})}(1)$ induces a canonical map

$$\tau : M(Proj(\mathcal{E})) \rightarrow \mathbb{Q}(1)[2]$$

in $\mathbf{DM}^{eff}(k, \mathbb{Q})$ by corollary 5.5 below. Here we take $\mathbb{Q}(1)[2] := C_*(-1, \infty) = N(\underline{Hom}(\Delta^\bullet, \mathbb{Q}_{tr}(-1, \infty)))$ where N is the normalized chain complex and \underline{Hom} is the internal Hom (see [29, page 15-16] for $\mathbb{Q}_{tr}(-1, \infty)$ and C_*). As the complex $C_*(-1, \infty)$ is fibrant for the projective motivic model structure (see [5, Corollary 2.155]), τ is represented by a morphism

$$\tau' : N(\mathbb{Q}_{tr}(V_\bullet)) \rightarrow C_*(-1, \infty)$$

in $K(PST(k))$ where V_\bullet is the Čech complex associated to the atlas $a : V \rightarrow Proj(\mathcal{E})$. By the Dold-Kan correspondence we get a morphism

$$(3.1) \quad \tau' : \mathbb{Q}_{tr}(V_\bullet) \rightarrow \underline{Hom}(\Delta^\bullet, \mathbb{Q}_{tr}(-1, \infty))$$

in $\Delta^{op}(PST(k))$. Note that in simplicial degree zero, the induced map $\mathbb{Q}_{tr}(V) \rightarrow \mathbb{Q}_{tr}(-1, \infty)$ represents the class of $\mathcal{O}_{Proj(\mathcal{E}|_V)}(1)$. Using the commutativity of

$$\begin{array}{ccc} \mathbb{Q}_{tr}(V_i) & \longrightarrow & \underline{Hom}(\Delta^i, \mathbb{Q}_{tr}(-1, \infty)) \\ \downarrow & & \downarrow we \\ \mathbb{Q}_{tr}(V) & \longrightarrow & \mathbb{Q}_{tr}(-1, \infty) \end{array}$$

the upper horizontal morphism also represents the class of $\mathcal{O}_{Proj(\mathcal{E}|_{V_i})}(1)$ modulo the \mathbb{A}^1 -weak equivalence we .

The morphism of simplicial presheaves (3.1) induces a morphism

$$(\tau')^m : \mathbb{Q}_{tr}(\underbrace{V_\bullet \times \cdots \times V_\bullet}_{m \text{ times}}) \rightarrow \underline{\mathit{Hom}}(\underbrace{\Delta^\bullet \times \cdots \times \Delta^\bullet}_{m \text{ times}}, \mathbb{Q}_{tr}(-^1, \infty)^{\wedge m})$$

between multisimplicial presheaves with transfers for every positive integer m . The diagonals $\Delta^\bullet \rightarrow \text{diag}(\Delta^\bullet \times \cdots \times \Delta^\bullet)$ and $V_\bullet \rightarrow \text{diag}(V_\bullet \times \cdots \times V_\bullet)$ give a morphism

$$(\tau')^m : \mathbb{Q}_{tr}(V_\bullet) \rightarrow \underline{\mathit{Hom}}(\Delta^\bullet, \mathbb{Q}_{tr}(-^1, \infty)^{\wedge m}).$$

Moreover the morphism $\mathbb{Q}_{tr}(V_\bullet) \rightarrow \mathbb{Q}_{tr}(U_\bullet)$ gives a morphism

$$\sigma : \mathbb{Q}_{tr}(V_\bullet) \rightarrow \text{diag}\left(\bigoplus_{m=0}^n \underline{\mathit{Hom}}(\Delta^\bullet, \mathbb{Q}_{tr}(-^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_\bullet))\right)$$

of simplicial presheaves with transfers. Here U_\bullet is the associated Čech complex of $a : U \rightarrow F$.

In degree i the morphism σ coincides with the one from [29, Construction 15.10] modulo the \mathbb{A}^1 -weak equivalence

$$\bigoplus_{m=0}^n \underline{\mathit{Hom}}(\Delta^i, \mathbb{Q}_{tr}(-^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_i)) \rightarrow \bigoplus_{m=0}^n \mathbb{Q}_{tr}(-^1, \infty)^{\wedge m} \otimes \mathbb{Q}_{tr}(U_i).$$

It follows from [29, Theorem 15.12] that σ induces \mathbb{A}^1 -weak equivalence after passing to the normalized complex. This proves the theorem. \square

3.3. Motives of blow-ups. Let X be a k -scheme. Let $X' \rightarrow X$ be a blow-up with center Z and $Z' := Z \times_X X'$ be the exceptional divisor. Then [29, Theorem 13.26] can be rephrased as follows. (Recall that $\text{char}(k) = 0$.)

Theorem 3.5. *The following commutative diagram*

$$\begin{array}{ccc} tr(Z') & \longrightarrow & tr(X') \\ \downarrow & & \downarrow \\ tr(Z) & \longrightarrow & tr(X) \end{array}$$

is homotopy co-cartesian (with respect to the étale \mathbb{A}^1 -local model structure).

Proposition 3.6. *Let F be a smooth Deligne-Mumford stack and $Z \subset F$ be a smooth closed substack. Let $\text{Bl}_Z(F)$ be the blow-up of F with center Z and $E := Z \times_F \text{Bl}_Z(F)$ be the exceptional divisor. Then one has a canonical distinguished triangle of the form :*

$$M(E) \rightarrow M(Z) \oplus M(\text{Bl}_Z(F)) \rightarrow M(F) \rightarrow M(E)[1].$$

Proof. Let $a : U \rightarrow F$ be an atlas and let U_\bullet be the associated Čech complex. Then the following square of simplicial presheaves with transfers

$$\begin{array}{ccc} \mathbb{Q}_{tr}(U_{\bullet} \times_F E) & \longrightarrow & \mathbb{Q}_{tr}(U_{\bullet} \times_F Bl_Z(F)) \\ \downarrow & & \downarrow \\ \mathbb{Q}_{tr}(U_{\bullet} \times_F Z) & \longrightarrow & \mathbb{Q}_{tr}(U_{\bullet}) \end{array}$$

is homotopy co-cartesian in each degree by theorem 3.5. Since homotopy colimits commutes with homotopy push-outs, the following square

$$\begin{array}{ccc} M(E) & \longrightarrow & M(Bl_Z(F)) \\ \downarrow & & \downarrow \\ M(Z) & \longrightarrow & M(F) \end{array}$$

is homotopy co-cartesian and hence we get our result. \square

Theorem 3.7. *Let F be a smooth Deligne-Mumford stack and $Z \subset F$ be a smooth closed substack of pure codimension c . Let $Bl_Z(F)$ be the blow-up of F with center Z . Then*

$$M(Bl_Z(F)) \cong M(F) \bigoplus (\oplus_{i=1}^{c-1} M(Z)(i)[2i]).$$

Proof. By proposition 3.6 we have a canonical distinguished triangle

$$M(p^{-1}(Z)) \rightarrow M(Z) \oplus M(Bl_Z(F)) \rightarrow M(F) \rightarrow M(p^{-1}(Z))[1],$$

where $p : Bl_Z(F) \rightarrow F$ is the blow-up. Since Z is smooth, $p^{-1}(Z) \cong Proj(N_Z(F))$, where $N_Z(F)$ is the normal bundle. Hence using theorem 3.4, it is enough to show that the morphism $M(F) \rightarrow M(p^{-1}(Z))[1]$ is zero in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Let $q : Bl_{Z \times \{0\}}(F \times \mathbb{A}^1) \rightarrow F \times \mathbb{A}^1$ be the blow-up of $Z \times \{0\}$ in $F \times \mathbb{A}^1$.

Following the proof of [38, Proposition 3.5.3], consider the morphism of exact triangles:

$$\begin{array}{ccccc} M(p^{-1}(Z)) & \longrightarrow & M(q^{-1}(Z \times \{0\})) & & \\ \downarrow & & \downarrow & & \\ M(Z) \oplus M(Bl_Z F) & \longrightarrow & M(Z \times \{0\}) \oplus M(Bl_{Z \times \{0\}} F \times \mathbb{A}^1) & & \\ \downarrow & & \downarrow f & & \\ M(F) & \xrightarrow{s_0} & M(F \times \mathbb{A}^1) & & \\ \downarrow g & & \downarrow h & & \\ M(p^{-1}(Z))[1] & \xrightarrow{a} & M(q^{-1}(Z \times \{0\}))[1] & & \end{array}$$

Since the morphism s_0 is an isomorphism and since by theorem 3.4 a is split injective, the morphism g is zero if h is zero. To show that h is zero it is enough to show that f has a section. This is the case as the composition

$$M(F \times \{1\}) \rightarrow M(Bl_{Z \times \{0\}} F \times \mathbb{A}^1) \rightarrow M(F \times \mathbb{A}^1)$$

is an isomorphism. \square

3.4. Gysin triangle. Given a morphism $F \rightarrow F'$ of Deligne-Mumford stacks, let

$$M\left(\frac{F'}{F}\right) := \text{cone}(M(F) \rightarrow M(F')).$$

Similarly given a morphism $V_\bullet \rightarrow U_\bullet$ of simplicial schemes, let

$$\mathbb{Q}_{tr}\left(\frac{U_\bullet}{V_\bullet}\right) := \text{cone}(\mathbb{Q}_{tr}(V_\bullet) \rightarrow \mathbb{Q}_{tr}(U_\bullet)).$$

Lemma 3.8. *Let $f : F' \rightarrow F$ be an étale morphism of smooth Deligne-Mumford stacks, and let $Z \subset F$ be a closed substack such that f induces an isomorphism $f^{-1}(Z) \cong Z$. Then the canonical morphism*

$$M\left(\frac{F'}{F' - Z}\right) \rightarrow M\left(\frac{F}{F - Z}\right)$$

is an isomorphism.

Proof. Let $v' : V' \rightarrow F'$ be an atlas of F' , and let $v : V \rightarrow F - Z$ be an atlas of the complement of Z . Then $U = V \coprod V' \rightarrow F$ is an atlas of F . Let $f_\bullet : V'_\bullet \rightarrow U_\bullet$ be the induced morphism between the associated Čech simplicial schemes. In each simplicial degree i , we have an étale morphism $f_i : V'_i \rightarrow U_i$ such that f_i induces an isomorphism $Z \times_F V'_i \cong Z \times_F U_i$. Let $Z_\bullet := Z \times_F U_\bullet \cong Z \times_F V'_\bullet$. It is enough to show that the canonical morphism $M\left(\frac{V_\bullet}{V_\bullet - Z_\bullet}\right) \rightarrow M\left(\frac{U_\bullet}{U_\bullet - Z_\bullet}\right)$ is an isomorphism. This is indeed the case as $\mathbb{Q}_{tr}\left(\frac{V_\bullet}{V_\bullet - Z_\bullet}\right) \cong \mathbb{Q}_{tr}\left(\frac{U_\bullet}{U_\bullet - Z_\bullet}\right)$ by [37, Proposition 5.18]. \square

Lemma 3.9. *Let $p : V \rightarrow F$ be a vector bundle of rank d over a smooth Deligne-Mumford stack F . Let $s : F \rightarrow V$ be the zero section of p . Then*

$$M\left(\frac{V}{V \setminus s}\right) \cong M(F)(d)[2d].$$

Proof. Using lemma 3.8, we have an isomorphism

$$M\left(\frac{V}{V \setminus s}\right) \cong M\left(\frac{\text{Proj}(V \oplus O)}{\text{Proj}(V \oplus O) \setminus s}\right).$$

The image of the embedding $\text{Proj}(V) \rightarrow \text{Proj}(V \oplus O)$ is disjoint from s and $\iota : \text{Proj}(V) \rightarrow \text{Proj}(V \oplus O) \setminus s$ is the zero section of a line bundle. Thus, the induced morphism $\iota : M(\text{Proj}(V)) \rightarrow M(\text{Proj}(V \oplus O) \setminus s)$ is an \mathbb{A}^1 -weak equivalence. (This can be checked using an explicit \mathbb{A}^1 -homotopy as in the classical case where the base is a scheme.) It follows that

$$M\left(\frac{V}{V \setminus s}\right) \cong M\left(\frac{\text{Proj}(V \oplus O)}{\text{Proj}(V)}\right).$$

Now using theorem 3.4, we get

$$M\left(\frac{\text{Proj}(V \oplus O)}{\text{Proj}(V)}\right) \cong M(F)(d)[2d].$$

This proves the lemma. \square

Theorem 3.10. *Let $Z \subset F$ be a smooth closed codimension c substack of a smooth Deligne-Mumford stack F . Then there exists a Gysin exact triangle:*

$$M(F \setminus Z) \rightarrow M(F) \rightarrow M(Z)(c)[2c] \rightarrow M(F \setminus Z)[1].$$

Proof. We have the following obvious exact triangle

$$M(F \setminus Z) \xrightarrow{i} M(F) \rightarrow M\left(\frac{F}{F \setminus Z}\right) \rightarrow M(F \setminus Z)[1].$$

We need to show that $M\left(\frac{F}{F \setminus Z}\right) \cong M(Z)(c)[2c]$ in $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Let $D_Z(F)$ be the space of deformation to the normal cone and let $N_Z(F)$ be the normal bundle. Consider the following commutative diagram of stacks:

$$\begin{array}{ccccc} Z \times 1 & \longrightarrow & Z \times \mathbb{A}^1 & \longleftarrow & Z \times 0 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow s_Z \\ F \times 1 & \hookrightarrow & D_Z(F) & \hookleftarrow & N_Z(F) \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & 0. \end{array}$$

This gives morphisms

$$s^1 : M\left(\frac{F}{F \setminus Z}\right) \rightarrow M\left(\frac{D_Z(F)}{D_Z(F) \setminus (Z \times \mathbb{A}^1)}\right)$$

and

$$s^0 : M\left(\frac{N_Z(F)}{N_Z(F) \setminus s_Z}\right) \rightarrow M\left(\frac{D_Z(F)}{D_Z(F) \setminus (Z \times \mathbb{A}^1)}\right).$$

Let $U \rightarrow F$ be an atlas of F and let U_\bullet be the associated Čech simplicial scheme. Then s^1 can be described as

$$s^1 : \mathbb{Q}_{tr}\left(\frac{U_\bullet}{(F \setminus Z) \times_F U_\bullet}\right) \rightarrow \mathbb{Q}_{tr}\left(\frac{D_Z(F) \times_F U_\bullet}{(D_Z(F) \setminus (Z \times \mathbb{A}^1)) \times_F U_\bullet}\right).$$

Let $Z_i := Z \times_F U_i$. In each simplicial degree i the morphism $(s^1)_i : \mathbb{Q}_{tr}\left(\frac{U_i}{U_i \setminus (Z_i)}\right) \rightarrow \mathbb{Q}_{tr}\left(\frac{D_{Z_i}(U_i)}{D_{Z_i}(U_i) \setminus (Z_i \times \mathbb{A}^1)}\right)$ induced by s^1 is an \mathbb{A}^1 -weak equivalence by lemma 3.11. Hence s^1 is an \mathbb{A}^1 -weak equivalence. Similarly, s^0 is an \mathbb{A}^1 -weak equivalence. Hence we get an isomorphism $M\left(\frac{F}{F \setminus Z}\right) \cong M\left(\frac{N_Z(F)}{N_Z(F) \setminus s_Z}\right)$. But $M\left(\frac{N_Z(F)}{N_Z(F) \setminus s_Z}\right) \cong M(Z)(c)[2c]$ by lemma 3.9. \square

Lemma 3.11. *Suppose we have a cartesian diagram of smooth schemes*

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow 0 & & \downarrow u \\ \mathbb{A}^1 \times Z & \xrightarrow{v} & X \end{array}$$

where u and v are closed embeddings and Y has codimension 1 in X . Then the canonical morphism $M\left(\frac{Y}{Y \setminus Z}\right) \rightarrow M\left(\frac{X}{X \setminus (\mathbb{A}^1 \times Z)}\right)$ is an isomorphism.

Proof. Let c be the codimension of Z in Y ; it is also the codimension of $\mathbb{A}^1 \times Z$ in X . Using [38, Proposition 3.5.4] we have $M\left(\frac{Y}{Y \setminus Z}\right) \cong M(Z)(c)[2c]$ and $M\left(\frac{X}{X \setminus (\mathbb{A}^1 \times Z)}\right) \cong M(Z \times \mathbb{A}^1)(c)[2c]$. Since $M(Z) \cong M(Z \times \mathbb{A}^1)$, we get the lemma. \square

4. MOTIVES OF DELIGNE-MUMFORD STACKS, II

The main goal of this section is to show that the motive of a smooth Deligne-Mumford stack F is a direct factor of the motive of a smooth and quasi-projective variety. Moreover, if F is proper, this variety can be chosen to be projective.

4.1. Blowing-up Deligne-Mumford stacks and principalization. Let F be a smooth Deligne-Mumford stack and let $a : U \rightarrow F$ be an atlas of F . Let Z be a closed substack of F . The blow-up of F along Z is a Deligne-Mumford stack $Bl_Z F$ together with a representable projective morphism $\pi : Bl_Z F \rightarrow F$. The induced morphism $a' : Bl_{Z \times_F U} U \rightarrow Bl_Z F$ is an atlas. The existence of $Bl_Z F$ is a consequence of the fact that blow-ups commute with flat base change.

Theorem 4.1. *Let F be a smooth Deligne-Mumford stack of finite type over a field of characteristic zero. Let \mathcal{O}_F be the structure sheaf and let $\mathcal{I} \subset \mathcal{O}_F$ be a coherent ideal. Then there is a sequence of blow-ups in smooth centers*

$$\pi : F_r \xrightarrow{\pi_r} F_{r-1} \xrightarrow{\pi_{r-1}} \dots \xrightarrow{\pi_1} F$$

such that $\pi^* \mathcal{I} \subset \mathcal{O}_{F_r}$ is locally principal.

Proof. Let $a : U \rightarrow F$ be an atlas and denote $\mathcal{J} := a^* \mathcal{I}$. By Hironaka's resolution of singularities [25, Theorem 3.15], we have a sequence of blow-ups in smooth centers

$$\pi' : U_r \xrightarrow{\pi'_r} U_{r-1} \xrightarrow{\pi'_{r-1}} \dots \xrightarrow{\pi'_1} U,$$

such that $(\pi')^* \mathcal{J}$ is a locally principal coherent ideal on U_r . Moreover this sequence commutes with arbitrary smooth base change. Hence the sequence π' descends to give the sequence of the statement. \square

Lemma 4.2. *Let $F' \rightarrow F$ be a (quasi-)projective representable morphism of Deligne-Mumford stacks. Let X and X' be the coarse moduli spaces of F and F' respectively. Then the induced morphism $X' \rightarrow X$ is (quasi-)projective. In particular, if X is (quasi-)projective then so is X' .*

Proof. [27, lemma 2, Theorem 1]. \square

The proof of the following theorem was communicated to us by David Rydh.

Theorem 4.3. *Given a smooth finite type Deligne-Mumford stack F over k , there exists a sequence of blow-ups in smooth centers $\pi : F' \rightarrow F$, such that the coarse moduli space of F' is quasi-projective.*

Proof. Let $p : F \rightarrow X$ be the morphism to the coarse moduli space of F . X is a separated algebraic space. By Chow's Lemma ([24, Theorem 3.1]) we have a projective morphism $g : X' \rightarrow X$ from a quasi-projective scheme X' . Moreover by [18, Corollary 5.7.14] we may assume that g is a blow-up along a closed subspace $Z \subset X$. Let $F' := X' \times_X F$. There is a morphism $p' : F' \rightarrow X'$. Since F is tame X' is the coarse moduli space of F' ([1, Cor 3.3]). Let $T := Z \times_X F$ and let $\pi : Bl_T F \rightarrow F$ be the blow-up of F along T . Then $Bl_T F$ is the closure of $F \setminus T$ in F' . As F' is tame the coarse moduli space of $Bl_T F$ is a closed subscheme of X' . Hence $Bl_T F$ has quasi-projective coarse moduli space.

Now by 4.1 We have a sequence of blow-ups in smooth centers $\pi : F_r \rightarrow F$ such that the ideal sheaf defining T is principalized. Hence there exists a canonical projective representable morphism $\pi' : F_r \rightarrow Bl_T F$. Since $Bl_T F$ has quasi-projective coarse moduli space and π' is a projective representable morphism, we have our result by lemma 4.2. \square

4.2. Chow motives and motives of proper Deligne-Mumford stack.

Let $f : X \rightarrow Y$ be a finite morphism between smooth schemes such that each connected component of X maps surjectively to a connected component of Y and generically over Y the degree of f is constant equal to m . Then the transpose of Γ_f is a correspondence from Y to X . This defines a morphism ${}^t f : \mathbb{Q}_{tr}(Y) \rightarrow \mathbb{Q}_{tr}(X)$ such that $f \circ (\frac{1}{m} {}^t f)$ is the identity.

Remark 4.4. *Suppose we are given a cartesian diagram of smooth schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow h' & & \downarrow h \\ Y & \xrightarrow{f} & X \end{array}$$

with h étale. Assume that f is a finite morphism such that each connected component of Y maps surjectively to a connected component of X and generically over X the degree of f is constant equal to m . Then f' satisfies the same properties as f . Thus we have morphisms ${}^t f : \mathbb{Q}_{tr}(X) \rightarrow \mathbb{Q}_{tr}(Y)$ and ${}^t f' : \mathbb{Q}_{tr}(X') \rightarrow \mathbb{Q}_{tr}(Y')$. Using the definition of composition of finite correspondences one can easily verify that $(h') \circ ({}^t f') = ({}^t f) \circ (h)$.

Lemma 4.5. *Let F be a smooth Deligne-Mumford stack. Assume that there exists a smooth scheme X and a finite surjective morphism $g : X \rightarrow F$. Then $M(F)$ is a direct factor of $M(X)$.*

Proof. We may assume that F and X are connected. Let $a : U \rightarrow F$ be an atlas and U_\bullet the associated Čech complex. Set $V_\bullet := U_\bullet \times_F X$. Then $g'_\bullet : V_\bullet \rightarrow U_\bullet$ is finite and surjective of constant degree m in each simplicial

degree. It follows from 4.4 that ${}^t g'_\bullet : \mathbb{Q}_{tr}(U_\bullet) \rightarrow \mathbb{Q}_{tr}(V_\bullet)$ is a morphism of simplicial sheaves with transfers such that $g'_\bullet \circ (\frac{1}{m} {}^t g'_\bullet) = id$. Hence $\mathbb{Q}_{tr}(U_\bullet)$ is a direct factor of $\mathbb{Q}_{tr}(V_\bullet)$.

Since V_\bullet is a Čech resolution of X , we have $\mathbb{Q}_{tr}(V_\bullet) \cong \mathbb{Q}_{tr}(X)$ by [29, Proposition 6.12]. This proves the result. \square

Theorem 4.6. *Let F be a proper (resp. not necessarily proper) smooth Deligne-Mumford stack. Then $M(F)$ is a direct summand of the motive of a projective (resp. quasi-projective) variety.*

Proof. We can assume that F is connected. By 4.3 we get a sequence of blow-ups with smooth centers $\pi : F' \rightarrow F$ such that F' has (quasi)-projective coarse moduli space. By 3.7 $M(F)$ is a direct summand of $M(F')$. By [27, Theorem 1] there exists a smooth (quasi)-projective variety X and a finite flat morphism $g : X \rightarrow F'$. Hence $M(F')$ is a direct summand of $M(X)$ which proves our claim. \square

Recall that the category of effective geometric motives $\mathbf{DM}_{gm}^{eff}(k, \mathbb{Q})$ is the thick subcategory of $\mathbf{DM}^{eff}(k, \mathbb{Q})$ generated by the motives $M(X)$ for $X \in Sm/k$ (see [29, Definition 14.1]).

Corollary 4.7. *For any smooth finite type Deligne-Mumford stack F , $M(F)$ is an effective geometric motive.*

Remark 4.8. *By [29, Proposition 20.1] the category of effective Chow motives embeds into $\mathbf{DM}^{eff}(k, \mathbb{Q})$. Theorem 4.6 shows that $M(F)$ lies in the essential image of this embedding for any smooth proper Deligne-Mumford stack F .*

5. MOTIVIC COHOMOLOGY OF STACKS

Let F be a smooth Deligne-Mumford stack. For each integer i let $\mathbb{Q}(i) \in \mathbf{DM}^{eff}(k, \mathbb{Q})$ denote the motivic complex of weight i with rational coefficients (see [29, Definition 3.1]).

Definition 5.1. *The étale site $F_{\acute{e}t}$ is defined as follows. The objects of $F_{\acute{e}t}$ are couples (X, f) with X a scheme and $f : X \rightarrow F$ a representable étale morphism. A morphism from (X, f) to (Y, g) is a couple (ϕ, α) , where $\phi : X \rightarrow Y$ is a morphism of schemes and $\alpha : f \cong g \circ \phi$ is a 2-isomorphism. Covering families of an object (U, u) are defined as families $\{u_i : U_i \rightarrow U\}_{i \in I}$ such that the u_i 's are étale and $\bigcup u_i : \coprod_i U_i \rightarrow U$ is surjective.*

Definition 5.2. *The motivic cohomology of F with rational coefficients is defined as $H_M^{2i-n}(F, i) := H^{2i-n}(F_{\acute{e}t}, \mathbb{Q}(i)|_{F_{\acute{e}t}})$.*

Remark 5.3. *In [22, 3.0.2], motivic cohomology of an algebraic stack F is defined using the smooth site of F . For Deligne-Mumford stacks, this coincides with our definition by [22, Proposition 3.6.1(ii)].*

Lemma 5.4. *Let F be a Deligne-Mumford stack. We have an isomorphism*

$$H_M^{2i-n}(F, i) \simeq \text{Hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(M(F), \mathbb{Q}(n)[2i-n]).$$

Proof. Let $U \rightarrow F$ be an atlas and U_\bullet be the associated Čech complex. We have an étale weak equivalence $\mathbb{Q}(U_\bullet) \rightarrow \mathbb{Q}$ of complexes of sheaves on $F_{\text{ét}}$. Here \mathbb{Q} is the constant sheaf on $F_{\text{ét}}$. Writing $D(F_{\text{ét}})$ for the derived category of sheaves of \mathbb{Q} -vector spaces on $F_{\text{ét}}$, we thus have

$$H_M^{2i-n}(F, i) \cong \text{Hom}_{D(F_{\text{ét}})}(\mathbb{Q}(U_\bullet), \mathbb{Q}(i)[2i-n]).$$

Let $a : \mathbb{Q}(i) \rightarrow L$ be a fibrant replacement for the injective local model structure on $K(PST(k))$ and let $b : L|_{F_{\text{ét}}} \rightarrow M$ be a fibrant replacement for the injective local model structure on $K(F_{\text{ét}})$.

Since both a and b are étale local weak equivalences the composition $b \circ a : \mathbb{Q}(i)|_{F_{\text{ét}}} \rightarrow M$ is an étale weak equivalence. It follows that

$$\text{Hom}_{D(F_{\text{ét}})}(\mathbb{Q}(U_\bullet), (\mathbb{Q}(i)|_{F_{\text{ét}}})[2i-n]) \cong \text{Hom}_{Ho(K(F_{\text{ét}}))}(\mathbb{Q}(U_\bullet), M[2i-n]).$$

Using [39, 2.7.5], it follows that $H_M^{2i-n}(F, i)$ is the $(2i-n)$ -th cohomology of the complex $\text{Tot}(\text{Hom}(\mathbb{Q}(U_\bullet), M))$.

On the other hand, since a is an étale weak equivalence and $\mathbb{Q}(i)$ is \mathbb{A}^1 -local, L is also \mathbb{A}^1 -local. It follows that

$$\text{Hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(\mathbb{Q}_{tr}(U_\bullet), \mathbb{Q}(i)[2i-n]) \cong \text{Hom}_{Ho(K(PST(k)))}(\mathbb{Q}_{tr}(U_\bullet), L[2i-n]).$$

Again by ([39, 2.7.5]), the right hand side is same as $(2i-n)$ -th cohomology of the complex $\text{Tot}(\text{Hom}(\mathbb{Q}_{tr}(U_\bullet), L))$.

To prove the lemma it is now sufficient to show that $L(X) \rightarrow M(X)$ is a quasi-isomorphism for any smooth k -scheme X . By definition

$$H^n(L(X)) \cong H^n(\text{Hom}(\mathbb{Q}_{tr}(X), F)) \cong xt^n(\mathbb{Q}_{tr}(X), \mathbb{Q}(i)[n])$$

and by [29, 6.25] we have

$$xt^n(\mathbb{Q}_{tr}(X), \mathbb{Q}(i)[n]) = H_{\text{ét}}^n(X, \mathbb{Q}(i))$$

which is same as $H^n(M(X))$. □

Corollary 5.5. *Let F be a Deligne-Mumford stack and let \mathcal{O}_F be the structure sheaf. Then we have an isomorphism*

$$\text{Pic}(F) \otimes \mathbb{Q} \cong H_{\text{ét}}^1(F, \mathcal{O}_F^\times \otimes \mathbb{Q}) \cong \text{Hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(M(F), \mathbb{Q}(1)[2])$$

Proof. The first isomorphism follows from [31, page 65, 67]. By [29, Theorem 4.1] $\mathcal{O}_F^*[1] \otimes \mathbb{Q} \cong \mathbb{Q}(1)[2]$. So the second isomorphism is a particular case of lemma 5.4. □

Remark 5.6. *From the proofs, it is easy to see that Lemma 5.4 and Corollary 5.5 are true integrally if we use Voevodsky's category of étale motives with integral coefficients $\mathbf{DM}^{\text{eff}, \text{ét}}(k, \mathbb{Q})$.*

6. CHOW MOTIVES OF STACKS AND COMPARISONS

Let F be a smooth Deligne-Mumford stack.

Definition 6.1 ([22]). *The codimension m rational Chow group of F is defined to be*

$$A^m(F) := H_M^{2m}(F, m)_\mathbb{Q}.$$

Remark 6.2. *In [16, 34], the rational Chow groups are defined as the étale cohomology of suitable K -theory sheaves. This agrees with our definition by [22, Theorem 3.1, 5.3.10].*

Let \mathcal{M}_k (resp. $\mathcal{M}_k^{\text{eff}}$) be the category of covariant Chow motives (resp. effective Chow motives) with rational coefficients. The construction of the category of Chow motives for smooth and proper Deligne-Mumford stacks using the theory A^* was done in [7, §8]. We will denote the category of Chow motives (resp. effective Chow motives) for smooth and proper Deligne-Mumford stacks by $\mathcal{M}_k^{\text{DM}}$ (resp. $\mathcal{M}_k^{\text{DM, eff}}$).

Let C be a symmetric monoidal category and let $X \in \text{Ob}(C)$. Recall that an object $Y \in C$ is called a strong dual of X if there exist two morphisms $\text{coev} : \mathbb{1} \rightarrow Y \otimes X$ and $\text{ev} : X \otimes Y \rightarrow \mathbb{1}$, such that the composition of

$$(6.1) \quad X \xrightarrow{\text{id} \otimes \text{coev}} X \otimes Y \otimes X \xrightarrow{\text{ev} \otimes \text{id}} X$$

and the composition of

$$(6.2) \quad Y \xrightarrow{\text{coev} \otimes \text{id}} Y \otimes X \otimes Y \xrightarrow{\text{id} \otimes \text{ev}} Y$$

are identities.

Lemma 6.3. *Let F be a proper smooth Deligne-Mumford stack of pure dimension d . Then $h_{\text{DM}}(F) := (F, \Delta_F, 0)$ has a strong dual in $\mathcal{M}_k^{\text{DM}}$. It is given by $(F, \Delta_F, -d)$.*

Proof. Set $h_{\text{DM}}(F)^* := (F, \Delta_F, -d)$. We need to give morphisms $\text{coev} : \mathbb{1} \rightarrow h_{\text{DM}}(F)^* \otimes h_{\text{DM}}(F)$ and $\text{ev} : h_{\text{DM}}(F) \otimes h_{\text{DM}}(F)^* \rightarrow \mathbb{1}$, such that (6.1) and (6.2) are satisfied. The morphisms coev and ev are given by $\Delta_F \in A^d(F \times F)$. To compute the composition of (6.1), we observe that intersection of the cycles $\Delta_F \times \Delta_F \times F$ and $F \times \Delta_F \times \Delta_F$ in $F \times F \times F \times F \times F$ is equal to $\delta(F)$ where $\delta : F \rightarrow F \times F \times F \times F \times F$ is the diagonal morphism. The push-forward to $F \times F$ of the latter is simply the diagonal of $F \times F$. This shows that the composition of (6.1) is the identity of $h_{\text{DM}}(F)$. The composition of (6.2) is treated using the same method. \square

By [34, Theorem 2.1] the natural functor $e : \mathcal{M}_k \rightarrow \mathcal{M}_k^{\text{DM}}$ is an equivalence of \mathbb{Q} -linear tensor categories. This equivalence preserves the sub-categories of effective motives. Thus, after inverting this equivalence we can associate an effective Chow motive $h(F) \in \mathcal{M}_k^{\text{eff}}$ to every smooth and proper Deligne-Mumford stack F . On the other hand, by [29, Proposition 20.1] there exists a fully faithful functor $\iota : \mathcal{M}_k^{\text{eff}} \rightarrow \mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$.

Theorem 6.4. *Let F be a smooth proper Deligne-Mumford stack. Then $M(F) \cong \iota \circ h(F)$.*

Proof. We may assume that F has pure dimension d . By 4.6 $M(F)$ is a direct factor of the motive of a smooth and projective variety W such that $\dim(W) = d$. By [29, Example 20.11],

$$\underline{\text{Hom}}(M(W), \mathbb{Q}(d)[2d]) \cong M(W)$$

is an effective Chow motive. It follows that $\underline{\text{Hom}}(M(F), \mathbb{Q}(d)[2d])$ is also an effective Chow motive.

We first show that $\iota \circ h(F) \cong \underline{\text{Hom}}(M(F), \mathbb{Q}(d)[2d])$. Let \mathcal{V}_k be the category of smooth and projective varieties over k . For $M \in \mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ denote ω_M the presheaf on \mathcal{V}_k defined by

$$X \in \mathcal{V}_k \mapsto \text{Hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(M(X), M).$$

Using A.1, it is enough to construct an isomorphism of presheaves

$$\omega_{\underline{\text{Hom}}(M(F), \mathbb{Q}(d)[2d])} \cong \omega_{\iota \circ h(F)}.$$

The right hand side is by definition the presheaf $A^{\dim(F)}(- \times F)$. For $X \in \mathcal{V}_k$, we have

$$\begin{aligned} \omega_{\underline{\text{Hom}}(M(F), \mathbb{Q}(d)[2d])}(X) &= \text{hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(M(X), \underline{\text{Hom}}(M(F), \mathbb{Q}(d)[2d])) \\ &= \text{hom}_{\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})}(M(X \times F), \mathbb{Q}(d)[2d]) \\ &= H_M^{2d}(X \times F, d). \end{aligned}$$

We conclude using [22, Theorem 3.1(i) and Theorem 5.3.10].

To finish the proof, it remains to construct an isomorphism $\iota \circ h(F) \cong \underline{\text{Hom}}(\iota \circ h(F), \mathbb{Q}(d)[2d])$. It suffices to do so in the stable triangulated category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Q})$ in which $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ embeds fully faithfully by Voevodsky's cancellation theorem. (Recall that $\mathbf{DM}(k, \mathbb{Q})$ is defined as the homotopy category of $T = \mathbb{Q}_{tr}(\mathbb{A}^1/\mathbb{A}^1 - 0)$ -spectra for the stable motivic model structure; for more details, see [4, Définition 2.5.27] in the special case where the valuation on k is trivial.) In $\mathbf{DM}(k, \mathbb{Q})$, we have an isomorphism

$$\underline{\text{Hom}}(\iota \circ h(F), \mathbb{Q}(d)[2d]) \simeq \underline{\text{Hom}}(\iota \circ h(F), \mathbb{Q}(0)) \otimes \mathbb{Q}(d)[2d].$$

As the full embedding $\mathcal{M}_k \rightarrow \mathbf{DM}(k, \mathbb{Q})$ and the equivalence $\mathcal{M}_k \simeq \mathcal{M}_k^{\text{DM}}$ are tensorial, they preserve strong duals. From Lemma 6.3, it follows that $\underline{\text{Hom}}(\iota \circ h(F), \mathbb{Q}(0))$ is canonically isomorphic to $\iota(F, \Delta_F, -d) = \iota \circ h(F) \otimes \mathbb{Q}(-d)[-2d]$. This gives the isomorphism $\underline{\text{Hom}}(\iota \circ h(F), \mathbb{Q}(d)[2d]) \simeq \iota \circ h(F)$ we want. \square

APPENDIX A.

As usual, we fix a base field k of characteristic 0. (Varieties will be always defined over k .) Recall that $\mathcal{M}_k^{\text{eff}}$ is the category of effective Chow motives with rational coefficients. We will have to consider the following categories of varieties.

- (1) \mathcal{V}_k : the category of smooth and projective varieties.
- (2) \mathcal{V}'_k : the category of projective varieties having at most global quotient singularities, i.e., those that can be written as a quotient of an object of \mathcal{V}_k by a finite group.
- (3) \mathcal{N}_k : the category of projective normal varieties.
- (4) \mathcal{P}_k : the category of all projective varieties.

We have the chain of inclusions

$$\mathcal{V}_k \subset \mathcal{V}'_k \subset \mathcal{N}_k \subset \mathcal{P}_k.$$

Given $N \in \mathcal{M}_k^{\text{eff}}$ we define a functor $\omega_N : \mathcal{V}_k^{\text{op}} \rightarrow \text{Vec}_{\mathbb{Q}}$ by

$$\omega_N(X) = \text{Hom}_{\mathcal{M}_k^{\text{eff}}}(M(X), N), \text{ for } X \in \mathcal{V}_k.$$

We thus have a functor $\omega : \mathcal{M}_k^{\text{eff}} \rightarrow \text{PSh}(\mathcal{V}_k)$ given by $N \mapsto \omega_N$.

Theorem A.1. *The functor $\omega : \mathcal{M}_k^{\text{eff}} \rightarrow \text{PSh}(\mathcal{V}_k)$ is fully faithful, i.e., for every $M, N \in \mathcal{M}_k^{\text{eff}}$, the natural morphism*

$$(A.1) \quad \text{Hom}_{\mathcal{M}_k^{\text{eff}}}(M, N) \rightarrow \text{Hom}(\omega_M, \omega_N)$$

is bijective.

Remark A.2. *The statement of the theorem appears without proof in [32, 2.2] and is also mentioned in [34, p. 12].*

Lemma A.3. *The functor ω is faithful.*

Proof. To show that the map (A.1) is injective, we may assume that $M = M(X)$ and $N = M(Y)$ for $X, Y \in \mathcal{V}_k$. In this case, (A.1) has a retraction given by $\alpha \in \text{Hom}(\omega_M, \omega_N) \mapsto \alpha(id_X)$. Hence it is injective. \square

Definition A.4.

- (1) *The pcdh topology on \mathcal{P}_k is the Grothendieck topology generated by the covering families of the form $(X' \xrightarrow{p_{X'}} X, Z \xrightarrow{p_Z} X)$ such that $p_{X'}$ is a proper morphism, p_Z is a closed embedding and $p_{X'}^{-1}(X - p_Z(Z)) \rightarrow X - p_Z(Z)$ is an isomorphism. To avoid problems, we also add the empty family to the covers of the empty scheme.*
- (2) *The fh topology on \mathcal{N}_k is the topology associated to the pretopology formed by the finite families $(f_i : Y_i \rightarrow X)_{i \in I}$ such that $\cup_i f_i : \coprod_{i \in I} Y_i \rightarrow X$ is finite and surjective.*

Lemma A.5. *Let $M \in \mathcal{M}_k^{\text{eff}}$. The presheaf ω_M can be extended to a presheaf ω'_M on \mathcal{V}'_k such that for $X = X'/G$ with $X' \in \mathcal{V}_k$ and G a finite group, we have $\omega'_M(X) = \omega_M(X')^G$.*

Proof. By [14, Example 8.3.12], we can define refined intersection class with rational coefficients which can be used to define a category of effective Chow motives $\mathcal{M}'_k^{\text{eff}}$. Moreover, the canonical functor $\phi : \mathcal{M}_k^{\text{eff}} \rightarrow \mathcal{M}'_k^{\text{eff}}$, induced

by the inclusion $\mathcal{V}_k \rightarrow \mathcal{V}'_k$, is an equivalence of categories (cf. [6, Proposition 1.2]). For $X \in \mathcal{V}'_k$, we set

$$\omega'_M(X) = \text{Hom}_{\mathcal{M}'^{\text{eff}}_k}(M(X), \phi(M)).$$

In this way we get a presheaf ω'_M on \mathcal{V}'_k which extends the presheaf ω_M . Moreover, the identification $\omega'_M(X'/G) = \omega_M(X')^G$ is clear. \square

Lemma A.6. *Let $M \in \mathcal{M}'^{\text{eff}}_k$. The presheaf ω_M can be uniquely extended to a pcdh-sheaf ω''_M on \mathcal{P}_k .*

Proof. From B.2(1) and the blow-up formula for Chow groups we deduce that ω_M is a pcdh-sheaf on \mathcal{V}_k . The result now follows from the first claim in B.2. \square

Lemma A.7. *Let $M \in \mathcal{M}'^{\text{eff}}_k$. We have $\omega''_M|_{\mathcal{V}'_k} \cong \omega'_M$.*

Proof. We will show that ω'_M extends uniquely to a pcdh-sheaf on \mathcal{P}_k . Since $\omega'_M|_{\mathcal{V}_k} \cong \omega_M$, A.6 shows that this extension is given by ω''_M . In particular, we have $\omega''_M|_{\mathcal{V}'_k} \cong \omega'_M$.

From the first statement in B.2, it suffices to show that ω'_M is a pcdh-sheaf on \mathcal{V}'_k . To do so, we use B.2(2). Let $X \in \mathcal{V}_k$ and G a finite group acting on X . Let $Z \subset X$ be a smooth closed subscheme globally invariant under G . Let \tilde{X} be the blow-up of X along Z and let E be the exceptional divisor. We need to show that

$$\omega'_M(X/G) \simeq \ker\{\omega'_M(\tilde{X}/G) \oplus \omega'_M(Z/G) \rightarrow \omega'_M(E/G)\}.$$

This is equivalent to

$$\omega_M(X)^G \simeq \ker\{\omega_M(\tilde{X})^G \oplus \omega_M(Z)^G \rightarrow \omega_M(E)^G\}.$$

This is true by the blow-up formula for Chow groups and the exactness of the functor $(-)^G$ on $\mathbb{Q}[G]$ -modules. \square

Lemma A.8. *Let $M \in \mathcal{M}'^{\text{eff}}_k$. Then $\omega''_M|_{\mathcal{N}_k}$ is an fh-sheaf.*

Proof. Let $X = Y/G$ with $Y \in \mathcal{N}_k$ and G a finite group. We claim that $\omega''_M(Y)^G \cong \omega''_M(X)$. When Y is smooth, this is true by A.5 and A.7. In general, we will prove this by induction on the dimension of Y and we will no longer assume that Y is normal. (However, it is convenient to assume that Y is reduced.) If Y has dimension zero then Y is smooth and the result is known. Assume that $\dim(Y) = d > 0$. By G -equivariant resolution of singularities there is a blow-up square

$$\begin{array}{ccc} E & \xrightarrow{g} & Y' \\ \downarrow h & & \downarrow f \\ Z & \xrightarrow{i} & Y \end{array}$$

such that Y' is smooth, $Z \subset Y$ is a nowhere dense closed subscheme which is invariant under the action of G and such that $Y' - E \simeq Y - Z$. Taking quotients by G gives the following blow-up square

$$\begin{array}{ccc} E/G & \longrightarrow & Y'/G \\ \downarrow & & \downarrow \\ Z/G & \longrightarrow & Y/G. \end{array}$$

Using induction on dimension and the fact that ω_M'' is a *pcdh*-sheaf, we are left to show that $\omega_M''(Y')^G \simeq \omega_M''(Y'/G)$. This follows from A.5 and A.7. \square

Proof of Theorem A.1. It remains to show that the functor is full. Let $M, N \in \mathcal{M}_k^{\text{eff}}$. Since every effective motive is a direct summand of the motive of a smooth and projective variety, we may assume that $M = M(X)$ and $N = M(Y)$ for $X, Y \in \mathcal{V}_k$. Let $f : \omega_{M(X)} \rightarrow \omega_{M(Y)}$ be a morphism of presheaves. As in the proof of A.3, there is an associated morphism of Chow motives $f_X(id_{M(X)}) : M(X) \rightarrow M(Y)$. For $Z \in \mathcal{V}_k$ and $c \in \omega_{M(X)}(Z)$, we need to show that $f_X(id_{M(X)}) \circ c = f_Z(c) \in \omega_{M(Y)}(Z)$.

By A.7 the morphism f can be uniquely extended to a morphism $f'' : \omega_{M(X)}'' \rightarrow \omega_{M(Y)}''$ of *pcdh*-sheaves on \mathcal{P}_k . Moreover, by A.8, the restriction of f'' to \mathcal{N}_k is a morphism of *fh*-sheaves. By [4, Prop 2.2.6] (see also [33]), any *fh*-sheaf has canonical transfers and $f''|_{\mathcal{N}_k}$ commutes with them. Now $c \in \omega_{M(X)}(Z) = CH^*(Z \times X)$ is the class of a finite correspondence $\gamma \in Cor(Z, X)$ and $c = \omega'_{M(X)}(\gamma)(id_{M(X)})$. (This follows from [29, Corollary 19.2] and the property that $C_*\mathbb{Q}_{tr}(X)$ is fibrant with respect to the projective motivic model structure; this property holds because X is proper, see [5, Cor. 1.1.8].) Thus, we have:

$$f_Z(c) = f'_Z(\omega'_{M(X)}(\gamma)(id_{M(X)})) = \omega'_{M(Y)}(\gamma)(f'_X(id_{M(X)})) = f_X(id_{M(X)}) \circ c.$$

This completes the proof. \square

APPENDIX B.

Let C' be a category and τ' a Grothendieck topology on C' . Given a functor $u : C \rightarrow C'$ there is an induced topology τ on C . (For the definition of the induced topology, we refer the reader to [3, III 3.1].)

Proposition B.1. *Assume that $u : C \hookrightarrow C'$ is fully faithful and that every object of C' can be covered, with respect to the topology τ' , by objects in $u(C)$. Let $X \in C$ and $R \subset X$ be a sub-presheaf of X . Then, the following conditions are equivalent:*

- (1) $R \subset X$ is a covering sieve for τ .
- (2) There exists a family $(X_i \rightarrow X)_{i \in I}$ such that
 - (a) $R \supset \text{Image}(\coprod_i X_i \rightarrow X)$;
 - (b) $(u(X_i) \rightarrow u(X))_{i \in I}$ is a covering family for τ' .

Moreover $u_* : \text{Shv}(C') \rightarrow \text{Shv}(C)$ is an equivalence of categories.

Proof. The last assertion is just [3, Théorème III.4.1].

(1) \Rightarrow (2): Suppose $R \subset X$ is a covering sieve for τ . Then $u^*(R) \rightarrow u(X)$ is a bicoverying morphism for τ' , i.e., induces an isomorphism on the associated sheaves (see [3, Définition I.5.1, Définition II.5.2 and Proposition III. 1.2] where u^* was denoted by $u_!$ which is not so standard nowadays). Since C contains a generating set of objects for τ' , there is a covering family of the form $(u(X_i) \rightarrow u(X))_{i \in I}$ for the topology τ' and a dotted arrow as below

$$\begin{array}{ccc} & u^*(R) & \\ \nearrow & \downarrow & \\ \coprod_i u(X_i) & \longrightarrow & u(X) \end{array}$$

making the triangle commutative.

Now, recall that for $U' \in C'$, one has

$$u^*(R)(U') = \underset{(V, U' \rightarrow u(V)) \in U' \setminus C}{\operatorname{colim}} R(V)$$

where $U' \setminus C$ is the comma category. Using the fact that $u : C \hookrightarrow C'$ is fully faithful, we see that for $U' = u(U)$ the category $u(U) \setminus C$ has an initial object given by $(U, id : u(U) = u(U))$. It follows that $R(U) \simeq u^*(R)(u(U))$ which can be also written as $R \simeq u_* u^*(R)$. In particular, the maps of presheaves $u(X_i) \rightarrow u^*(R)$ are uniquely induced by maps of presheaves $X_i \rightarrow R$. This shows that R contains the image of the morphism of presheaves $\coprod_i X_i \rightarrow X$.

(2) \Rightarrow (1): Now suppose that condition (2) is satisfied. We must show that $u^*(R) \rightarrow u(X)$ is a bicoverying morphism of presheaves for τ' , i.e., that $a_{\tau'}(u^*(R)) \rightarrow a_{\tau'}(u(X))$ is an isomorphism where $a_{\tau'}$ is the “associated τ' -sheaf” functor.

Since the surjective morphism of sheaves $a_{\tau'}(\coprod_i u(X_i)) \rightarrow a_{\tau'}(u(X))$ factors through $a_{\tau'}(u^*(R))$, the surjectivity of $a_{\tau'}(u^*(R)) \rightarrow a_{\tau'}(u(X))$ is clear. Since every object of C' can be covered by objects in $u(C)$, to prove injectivity it suffices to show that $u^*(R)(u(U)) \rightarrow u(X)(u(U))$ is injective for all $u \in C$. From the proof of the implication (1) \Rightarrow (2), we know that this map is nothing but the inclusion $R(U) \hookrightarrow X(U)$. This finishes the proof. \square

The $pcdh$ topology on \mathcal{P}_k induces topologies on \mathcal{V}_k and \mathcal{V}'_k which we also call $pcdh$. The next corollary gives a description of these topologies.

Corollary B.2. *The categories of $pcdh$ -sheaves on \mathcal{V}_k and \mathcal{V}'_k are equivalent to the category of $pcdh$ -sheaves on \mathcal{P}_k . Moreover, $pcdh$ -sheaves on \mathcal{V}_k and \mathcal{V}'_k can be characterized as follows.*

- (1) *A presheaf F on \mathcal{V}_k such that $F(\emptyset) = 0$ is a $pcdh$ -sheaf if and only if for every smooth and projective variety X , and every closed and smooth subscheme $Z \subset X$, one has*

$$F(X) \simeq \ker\{F(\tilde{X}) \oplus F(Z) \rightarrow F(E)\}$$

where \tilde{X} is the blow-up of X in Z and $E \subset \tilde{X}$ is the exceptional divisor.

(2) A presheaf F on \mathcal{V}'_k such that $F(\emptyset) = 0$ is a pcdh-sheaf if and only if for every smooth and projective variety X together with an action of a finite group G , and every closed and smooth subscheme $Z \subset X$ globally invariant under the action of G , one has

$$F(X/G) \simeq \ker\{F(\tilde{X}/G) \oplus F(Z/G) \rightarrow F(E/G)\}$$

where \tilde{X} is the blow-up of X in Z and $E \subset \tilde{X}$ is the exceptional divisor.

Proof. By Hironaka's resolution of singularities, every projective variety can be covered (with respect to the pcdh topology) by smooth and projective varieties, i.e., by objects in the subcategory \mathcal{V}_k (and hence \mathcal{V}'_k). Thus, the first claim follows from [3, Théorème III.4.1].

Next, we only treat (2) as the verification of (1) is similar and in fact easier. The condition in (2) is necessary for F to be a pcdh-sheaf as $(\tilde{X}/G \rightarrow X/G, Z/G \rightarrow X/G)$ is a pcdh-cover. Hence we only need to show that the condition is sufficient.

Let $X/G \in \mathcal{V}'_k$ where X is a smooth and projective variety and G is a finite group acting on X . It suffices to show that $F(X/G) \simeq \text{Colim}_{R \subset (X/G)} F(R)$ where $R \subset (X/G)$ varies among covering sieves for the pcdh-topology on \mathcal{V}'_k . We will prove a more precise statement namely: any covering sieve $R \subset (X/G)$ can be refined into a covering sieve $R' \subset (X/G)$ such that $F(X/G) \simeq F(R')$.

By B.1, there exists a pcdh-cover $(Y_i \rightarrow (X/G))_i$ with $Y_i \in \mathcal{V}'_k$ and such that $R \supset \text{Image}(\coprod_i Y_i \rightarrow (X/G))$. Using equivariant resolution of singularities, we may find a sequence of equivariant blow-ups in smooth centers $Z_i \subset X_i$:

$$X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

such that the covering family

$$(B.1) \quad (X_n/G \rightarrow X/G, Z_{n-1}/G \rightarrow X/G, \dots, Z_0/G \rightarrow X/G)$$

is a refinement of the sieve R . Using induction and the property satisfied by F from (2), we see that

$$(B.2) \quad F(X/G) \simeq \ker\{F(X_n/G) \oplus F(Z_{n-1}/G) \oplus \cdots \oplus F(Z_0/G)$$

$$\longrightarrow F(E_n/G) \oplus \cdots \oplus F(E_1/G)\}$$

where $E_i \subset X_i$ is the exceptional divisor of the blow-up with center Z_{i-1} . It is easy to deduce from (B.2) that $F(X/G) \simeq F(R')$ when $R' \subset (X/G)$ is the image of the covering family (B.1). \square

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